Equivalence and dynamic analysis for jointed trusses based on improved finite elements

Jing Zhang, Zongquan Deng, Hongwei Guo and Rongqiang Liu

Abstract
Deployable trusses that are widely applied in space missions utilize many hinged members to adapt to different geometrical shapes and work conditions. This paper presents two improved finite elements for a link and a bending beam to calculate dynamic characteristics of non-jointed and jointed trusses. First, the axial and transverse wave motion equations of a beam are used to get the shape functions of the link and the beading beam in their axial and transverse directions. Second, the dynamic matrices of stiffness and mass for a link and a bending beam are established on the basis of virtual work principle. The dynamic matrix and a theoretical equation are used to calculate the natural frequency of a cantilever bar. The comparison of the two results shows the effectiveness of the dynamic stiffness matrix. These improved elements are proved to be more accurate in calculating the responses of structures at high frequencies than common elements. Equivalent models of non-jointed and jointed structures are obtained on the basis of the displacement and strain energy equivalence of a truss unit. The result of the period truss confirms the accuracy of the equivalent model of the non-jointed truss expressed by dynamic matrices. The dynamic influences of cubic joints on a jointed structure are evaluated by giving the simulations of the equivalent jointed structure based on improved matrices. The results indicate that the natural frequencies of the jointed structure increase with the excitation force and the stiffness of joints. A relational function among the natural frequencies, joint stiffness, and excitation force is also presented.

Keywords
Deployable structures, equivalence, non-linear dynamics, finite element, wave propagation

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Introduction
Given the difficulty in developing launchers capable of sending large payloads and high volume, the supporting structure for a telescope or an antenna reflector is bound to fit within the available capacity, so that small stowed volume and mass of the structure are required. Deployable structures have the potential to offer long-distance and large-surface support for apparatuses in astrophysics missions. An ultralight deployable mast has been designed to provide a means to scale up solar sails to gain large dimension. The aperture of the deployable antenna for Engineering Test Satellite VIII has reached 13 m. Carbon fiber booms have been used for propelling a sail craft as a supporting structure to reduce the mass and enhance the strength of the deployable structure.

Large deployable structures (LDSs) are too flexible to get ground tested, so the dynamic modeling and analysis of such structures are the keys to the design of such space structures. Many scholars have declared that the non-linear dynamic problem caused by joints is complex and troublesome. They have done much research of LDSs, which mainly include multibody dynamic simulation and structural vibration analysis for deploying and deployed dynamics of structures, respectively. The common methods for multibody dynamic modeling are based on contact models of the joint and the elastic assumption of the link. Two types of contact models are used to model a joint with clearance. One is a discontinuous model which uses Hertz theory to provide the relationship between deformation and contact force. The other is a continuous model which uses a rigid connection in place of the contact of a joint. Additionally, the friction, impact and energy

School of Mechatronics Engineering, Harbin Institute of Technology, Harbin, China

Corresponding author:
Hongwei Guo, School of Mechatronics Engineering, Harbin Institute of Technology, 2 Yikuang Street, 2F-406, Harbin 150001, China. Email: ghwhw@163.com
dissipation are considered in the contact model. Deploying simulation takes into account the elastic of the link, while the structural vibration analysis of jointed trusses often considers the influence of joints and tension cables. Rotation freedoms caused by joints are considered in wave propagation analysis of such deployable structures. The describing function method and the harmonic expansion method applied to express the non-linear characteristics of a joint are used to calculate the vibration modes and the natural frequencies.18,19 The describing function method and the harmonic expansion method applied to express the non-linear characteristics of a joint are used to calculate the vibration modes and the natural frequencies.18,19 The describing function method and the harmonic expansion method applied to express the non-linear characteristics of a joint are used to calculate the vibration modes and the natural frequencies.18,19 The describing function method and the harmonic expansion method applied to express the non-linear characteristics of a joint are used to calculate the vibration modes and the natural frequencies.18,19 The describing function method and the harmonic expansion method applied to express the non-linear characteristics of a joint are used to calculate the vibration modes and the natural frequencies.18,19

It is difficult for the above methods to solve the dynamic problem of large jointed structures. Some researchers raised an equivalent method for a jointed truss, which is based on strain energy equivalence and reduced models.20–22 The equivalence of the jointed structure is based on the constant stiffness and mass matrices which are obtained by the shape function assumption and the virtual work principle. The accuracy of dynamic simulation by these elements is determined by the number of elements and the iteration step-size. Because the matrices of stiffness and mass are not related to excitation frequencies, an accuracy response of a truss in high excitation frequencies is hard to give.

Therefore, this paper presents the dynamic matrices of the stiffness and the mass of a link and a beam by solving the equations of the axial and flexural wave motions respectively. The dynamic stiffness is used to evaluate the accuracy of the improved element by analyzing the natural frequency of a link with one fixed end. Then the equivalent models of the non-jointed and jointed structures based on the matrices of stiffness and mass are put forward. The amplitude-frequency characteristics of the equivalent non-jointed and jointed structures are obtained. Using the dynamic stiffness and mass, the equivalent model of the jointed truss can also analyze larger structures with high excitation frequency. The relational function among the natural frequency, joint stiffness and excitation force is established to get the effects of the non-linear stiffness of joints on the deployable structure.

**Formulation of improved finite elements**

A deployable truss is assembled by many rods or beams that are connected by joints. The dynamic response of a jointed truss can be obtained by using the finite element method. First, the improved stiffness and mass matrices for a link used in finite element method are established by solving the longitudinal wave equation. Then the matrices of stiffness and mass for a beam are derived based on the transverse wave equation. The matrices can satisfy the dynamic analysis of the truss in high excitation frequency because they are the functions of the excitation frequency.

**The dynamic matrices of stiffness and mass for a link based on the axial wave propagation**

The axial displacement of the cross section of the link at \( x \) is represented by \( u(x, t) \). And it satisfies axial wave equation

\[
\frac{\partial}{\partial x} \left( E_A \frac{\partial u}{\partial x} \right) - \rho A \frac{\partial^2 u}{\partial t^2} = 0
\]  

where \( A, E \) and \( \rho \) are the cross section, Young’s modulus and density of the link, respectively (shown in Figure 1) and \( t \) is the time.

The Fourier-transformed axial displacement \( u(x, t) \) with respect to \( t \) can be expressed as

\[
\tilde{u}(x, \omega) = \int_{-\infty}^{\infty} u(x, t) e^{-i\omega t} dt
\]

where \( \omega \) is the excitation frequency.

The Fourier-transformed equation (1) can be given by

\[
\frac{d^2 \tilde{u}}{dx^2} + \frac{\omega^2}{c_1^2} \tilde{u} = 0
\]

where \( c_1 = \sqrt{E/\rho} \).

The Fourier-transformed axial force can be written as

\[
\tilde{F} = EA \frac{d \tilde{u}}{dx}
\]

The solution of the displacement \( \tilde{u} \) can be expressed as

\[
\tilde{u} = a_1(\omega) e^{ik_1 x} + d_1(\omega) e^{-ik_1 x}
\]

where \( k_1 = \omega/c_1, a_1(\omega) \) and \( d_1(\omega) \) are the unspecified amplitudes which are related to the excitation frequency.

Based on equation (5), the transformed displacements of the link at the left and right ends \( \tilde{u}\left(0, \omega\right) \) and \( \tilde{u}\left(l, \omega\right) \) can be given by

\[
\begin{bmatrix}
\tilde{u}_1 \\
\tilde{u}_2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
e^{ik_1 l} & e^{-ik_1 l}
\end{bmatrix}
\begin{bmatrix}
a_1(\omega) \\
d_1(\omega)
\end{bmatrix}
\]

**Figure 1.** Link element.
where \( l \) is the length of the link.

The coefficients \( a_1(\omega) \) and \( d_1(\omega) \) derived from equation (6) can be expressed as

\[
\begin{bmatrix}
  a_1(\omega) \\
  d_1(\omega)
\end{bmatrix} = \begin{bmatrix}
  (1 - e^{2ik_1l})^{-1} & -(e^{-i\omega l} - e^{i\omega l})^{-1} \\
  (1 - e^{-2ik_1l})^{-1} & (e^{-i\omega l} - e^{i\omega l})^{-1}
\end{bmatrix} \begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2
\end{bmatrix}
\]  

(7)

By substituting the expressions of \( a_1(\omega) \) and \( d_1(\omega) \) of equation (7) in equation (5), the transformed displacement of the link can be written as

\[
\tilde{u} = \begin{bmatrix}
  \sin(k_1(l-x))/\sin(k_1l) & \sin(k_1x)/\sin(k_1l)
\end{bmatrix}
\times
\begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2
\end{bmatrix}
\]  

(8)

The coefficient matrix of \( \begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2
\end{bmatrix}^T \) in equation (8) can be substituted by \( N_l \). Based on equation (8), the strain of the link can be given by

\[
\frac{d\tilde{u}}{dx} = N_l \begin{bmatrix}
  \tilde{u}_1 \\
  \tilde{u}_2
\end{bmatrix}
\]  

(9)

Using the virtual work principle, the virtual strain energy can be expressed as

\[
W = EA \int_0^l \frac{d\tilde{u}}{dx} \cdot \frac{d\tilde{u}}{dx} \, dx
\]  

(10)

According to equations (9) and (10), the stiffness matrix of the link can be expressed as

\[
K_{el} = EA \int_0^l N_l^T N_l \, dx
\]  

(11)

Because an angle between the axial direction of the link and the x-axis of the global coordinate exists, the dynamic stiffness matrix can be expanded as

\[
K_{el}^E = \begin{bmatrix}
  K_{el}(1, 1) & 0 & K_{el}(1, 2) & 0 \\
  0 & 0 & 0 & 0 \\
  K_{el}(2, 1) & 0 & K_{el}(2, 2) & 0
\end{bmatrix}
\]  

(12)

where \( K_{el}(1, 1) = 1/4\lambda \sin(2k_1l) + 2k_1l \), \( K_{el}(2, 2) = K_{el}(1, 1) \), \( K_{el}(2, 1) = -1/2\lambda (k_1l \cos(k_1l) + \sin(k_1l)) \), \( K_{el}(1, 2) = K_{el}(2, 1) \), \( e_k = EA \frac{k_1}{\sin^2(k_1l)} \).

The mass matrix can be expressed by the shape function

\[
M_{el} = \rho A \int_0^l N_l^T N_l \, dx
\]  

(13)

The mass matrix can also be expressed as

\[
M_{el} = \begin{bmatrix}
  M_{el}(1, 1) & M_{el}(1, 2) \\
  M_{el}(2, 1) & M_{el}(2, 2)
\end{bmatrix}
\]  

(14)

where

\[
M_{el}(1, 1) = c_m(-\cos(k_1l) \sin(k_1l) + k_1l),
M_{el}(2, 1) = c_m(-\cos(k_1l)k_1 + \sin(k_1l)),
M_{el}(2, 2) = M_{el}(1, 1),
\]

\[
c_m = \rho A \frac{1}{2k_1 \sin(k_1l)}.
\]

The mass matrix can also be expanded as

\[
M_{el}^E = \begin{bmatrix}
  M_{el}(1, 1) & 0 & M_{el}(1, 2) & 0 \\
  0 & M_{el}(1, 1) & 0 & M_{el}(1, 2) \\
  M_{el}(2, 1) & 0 & M_{el}(2, 2) & 0 \\
  0 & M_{el}(2, 1) & 0 & M_{el}(2, 2)
\end{bmatrix}
\]  

(15)

Using the virtual work principle and the new shape function of the link, the matrices of stiffness and mass are presented. Because the matrices are related to the excitation frequency, they can be used to analyze the dynamic response of trusses in high excitation frequencies.

The dynamic matrices of stiffness and mass for a beam based on transverse wave propagation

Considering the moment and shear on the beam, the transformed transverse wave equations of a beam can be expressed as

\[
EI \frac{d^2 \tilde{v}_b}{dx^2} + \kappa GA \frac{d^2 \tilde{v}_s}{dx^2} = -\rho \omega^2 \frac{d\tilde{v}_b}{dx}
\]

\[
\phi(x, \omega) = \frac{d}{dx} \tilde{v}_b + \tilde{v}_s
\]

\[
\frac{\kappa GA}{dx^2} - \rho A \omega^2 (\tilde{v}_b + \tilde{v}_s)
\]  

(16)

where \( I \) is the inertia moment. \( \tilde{v}_b \) and \( \tilde{v}_s \) are the transformed transverse displacements respectively produced by bending and shear. \( \kappa \) is the shear coefficient of the beam. \( G \) is the shear modulus. \( \phi \), \( Q \), and \( M \) are the transformed transverse rotation angle of the cross section, the shear force, and the moment.
By solving equation (16), the transformed transverse displacement caused by bending and shear can be expressed as

$$
\ddot{v}_b(x, \omega) = a_2(\omega)e^{ik_2x} + d_2(\omega)e^{-ik_2x} + a_3(\omega)e^{ik_3x} + d_3(\omega)e^{-ik_3x}
$$

(17)

$$
\ddot{v}_s(x, \omega) = a_2a_2(\omega)e^{ik_2x} + a_2d_2(\omega)e^{-ik_2x} + a_3a_3(\omega)e^{ik_3x} + a_3d_3(\omega)e^{-ik_3x}
$$

(18)

where

$$
a_{2,3} = \frac{\rho\alpha^2}{\kappa Gk_{2,3}^2 - \rho\omega^2},
$$

$$
k_{2,3} = \sqrt{\frac{\omega^2 + \eta + \sqrt{(1 + \eta)^2 - 4(\eta - \omega^2/R_s^2)}}{-2\omega}},
$$

$$
\eta = \frac{E}{\kappa G}, \quad \text{and} \quad R = \frac{\ell}{\ell_A}.
$$

Because $a_{2,3} \ll 1$, it is assumed that the total transverse displacement of the beam $\ddot{v} = \ddot{v}_b$. Using equation (17), the total displacement of the beam can also be expressed as

$$
\ddot{v}(x, \omega) = \begin{bmatrix} e^{ik_2x} & e^{-ik_2x} & e^{ik_3x} & e^{-ik_3x} \end{bmatrix} C
$$

(19)

where $C = \begin{bmatrix} a_2(\omega) & a_2(\omega) & a_3(\omega) & a_3(\omega) \end{bmatrix}^T$.

The transverse displacements at the beam ends ($x = 0$ and $x = l$) can be obtained by equation (19). Based on the relationship between the transformed rotation angle and the transformed transverse displacement, $\phi(x, \omega) = d \ddot{v}_b/dx$, the rotation angles of the beam ends ($x = 0$ and $x = l$) can also be obtained. The transformed displacement vector of the beam can be written as

$$
\ddot{V} = \begin{bmatrix} \ddot{v}_1 & \ddot{\theta}_1 & \ddot{v}_2 & \ddot{\theta}_2 \end{bmatrix}^T
$$

(20)

The relation matrix of $V$ and $C$ can be written as $N_b$. The bending displacement coefficients in equation (20) can be expressed as

$$
C = N_b^{-1} V
$$

(21)

The expression of $N_b^{-1}$ is shown in Appendix 2. Based on equations (19) and (21), the transformed transverse displacement $\ddot{v}$ can be expressed as

$$
\ddot{v}(x, \omega) = \begin{bmatrix} e^{ik_2x} & e^{-ik_2x} & e^{ik_3x} & e^{-ik_3x} \end{bmatrix} N_b^{-1} V
$$

(22)

The coefficient vector of shape function of the beam can be written as

$$
N_b = \begin{bmatrix} e^{ik_2x} & e^{-ik_2x} & e^{ik_3x} & e^{-ik_3x} \end{bmatrix} N_b^{-1}
$$

(23)

The strain $\ddot{v}/dx^2$ can be given by

$$
\frac{d^2 \ddot{v}}{dx^2} = \ddot{N}_b V
$$

(24)

Using the virtual work principle, the virtual strain energy can be expressed. Therefore, the stiffness and mass matrices of beam can be given by

$$
K_{eb} = EI \int_0^l \ddot{N}_b^T \ddot{N}_b dx
$$

(25)

$$
M_{eb} = \rho A \int_0^l N_b^T N_b dx
$$

(26)

The established matrices of stiffness and mass for a link and a beam can be used to calculate the natural frequency of a link, a beam or a truss. Because the matrices of the link and the beam are related to the excitation frequency, the analysis results of the structures at high excitation frequencies are exact.

**Equivalence of non-jointed truss**

*Equivalent method for a truss based on the improved link element*

Trusses characterized by large size and periodicity can satisfy the application, the design, and the manufacture requirements of space structures in space missions. It is necessary to give an equivalent model for large trusses to reduce the complex calculation. By reducing the freedom of the finite element matrix, the unit of the truss shown in Figure 2 can be simplified into a link or a beam shown in Figures 1 and 3. The displacement relationship between the equivalent link and the truss unit is constant. The strain energy of the equivalent link is equal to the periodic unit of the truss.23,26

![Figure 2. Beam element.](image-url)
The relationship between the truss unit displacement $q$ and the node displacement of the equivalent link $v$ can be given by

$$
\begin{bmatrix}
u_1 \\
v_1 \\
u_2 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{7} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
q_6 \\
q_7 \\
q_8
\end{bmatrix} = B_e q
$$

(27)

The strain energy of the truss unit and the equivalent link can be written as

$$
U_T = \frac{1}{2} q^T K_T q, \quad U_{EL} = \frac{1}{2} l v^T K_{EL} v,
$$

(28)

where $K_T$ and $K_{EL}$ are the improved stiffness matrices of the truss unit and equivalent link.

Based on the equivalence of the truss unit energy $U_T = U_{EL}$, equation (27), and equation (28), the stiffness matrix of equivalent element can be given by

$$
K_{EL} = \frac{1}{l} (B_e B_e^T)^{-1} B_e K_T B_e^T (B_e B_e^T)^{-1}
$$

(29)

The kinetic energy of the truss unit $V_T$ and the equivalent element $V_{EL}$ can be expressed as

$$
V_T = \frac{1}{2} q^T M_T q, \quad V_{EL} = \frac{1}{2} l v^T M_{EL} v,
$$

(30)

Because the kinetic energy of the truss unit is equal to the equivalent element $V_T = V_{EL}$, the equivalent mass matrix can be written as

$$
M_{EL} = \frac{1}{l} (B_e B_e^T)^{-1} B_e M_T B_e^T (B_e B_e^T)^{-1}
$$

(31)

where $M_T$ is the mass matrix of the truss unit.

**Equivalent method for truss based on the improved beam element**

The method and assumption for the equivalence of a truss unit in lateral direction are the same as those in the previous section. The degrees of the freedom for the truss unit and an equivalent beam are shown in Figures 3 and 4. The equivalent beam element is based on improved beam element.

The displacement of the equivalent beam element can be expressed by the node displacement of the truss unit

$$
\begin{bmatrix}
v_1 \\
v_2 \\
\theta_1 \\
\theta_2
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
q_6 \\
q_7 \\
q_8
\end{bmatrix} = B_e q
$$

(32)

The stiffness and mass matrices of the equivalent beam can be given by substituting $B_e$ of equation (32) into equations (29) and (31).

**Equivalence of jointed truss**

A deployable structure consists of many links and joints. Due to the clearance of the joint, the relationship between the joint displacement and the force applied on the joint is non-linear, which affects the performance of the deployable structure. Thus, the...
non-linear characteristic of the joint should be considered in the dynamic simulation to get more accurate dynamic performance. Based on the dynamic matrices of stiffness and mass for the link, the equivalence of jointed truss unit is proposed. One link with two joints is shown in Figure 5, where \( d_1 \) and \( d_2 \) are the respective displacements of the joint and the link.\(^{26}\)

The applied force \( F_{NL} \) on the link is equal to the force on the joint. It is given by

\[
F_{NL} = K_1 d_1 + K_3 d_1^3
\]

(33)

where \( K_1 \) and \( K_3 \) are the coefficients of the linear and non-linear displacement of the joint.

The displacement of the joint in the link is expressed as

\[
d_1 = \left( \frac{K_3 d_1}{2 K_3} + \left( \frac{K_3 d_1}{2 K_3} + \frac{K_1 + 2 K_L}{3 K_3} \right)^{1/2} \right)^{1/3}
+ \left( \frac{K_3 d_1}{2 K_3} - \left( \frac{K_3 d_1}{2 K_3} + \frac{K_1 + 2 K_L}{3 K_3} \right)^{1/2} \right)^{1/3}
\]

(34)

where \( K_L \) is the stiffness of the link, \( d \) is the deformation of the jointed link, \( d = 2d_1 + d_2 \).

The harmonic expression of the displacement of the jointed link is given by \( d = a \sin \omega t + b \cos \omega t \). Based on the describing function method, the non-linear force can also be expressed as\(^{20}\)

\[
F_{NL} = c_d d = c_d (a \sin \omega t + b \cos \omega t) = c_d A_d \sin \varphi
\]

(35)

where \( A_d \) is the amplitude of \( d, A_d = \sqrt{a^2 + b^2} \), and \( c_d \) is the stiffness of jointed link.

The stiffness of jointed link \( c_d \) can be given by Fourier transforming

\[
c_d = \frac{2}{\pi A} \int_{0}^{\pi} (K_1 d_1 + K_3 d_1^3) \sin \varphi \, d\varphi
\]

(36)

can be obtained by numerical integration.

Two joints exist in each link of the truss unit as shown in Figure 2. The displacements of the equivalent beam can be expressed by the displacements of the truss unit

\[
\begin{bmatrix}
u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\
\end{bmatrix} = B \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}^{-1}
\]

(37)

The displacements \( q_1, q_3, q_5, \) and \( q_7 \) of \( q \) can be calculated by equation (37) when \( u_1, \theta_1, u_2, \) and \( \theta_2 \) are known. Because the stiffness of the link \( e_1 \) as shown in Figure 2 has little influence on the equivalent beam stiffness matrix, the constraints in \( y \) direction can be expressed as

\[
\begin{bmatrix}
u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\
\end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ -1/l & 0 & 1/l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & -1/l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/l \\
\end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}
\]

(38)

The iteration process for the calculation of large trusses is shown in Figure 6. The dynamic characteristics can be obtained by using the following steps:

- The values of the truss parameters and the initial displacements of the equivalent beams \( r \) are set.
- The node displacement vector of each truss unit \( q \) is calculated by equations (37) and (38) which contain eight linear independent equations.
- The deformation of the jointed link \( d \) can be calculated on the basis of \( q \) obtained in step 2. Then, the displacement of the joint \( d_1 \) is computed by equation (34), where \( K_L \) equals to the dynamics stiffness \( K_{dP}(1,1) \) or \( K_{dS}(1,2) \) in equation (12). Substituting \( d_1 \) in equation (36), the stiffness of jointed link \( c_d \) is obtained by integral.
- Base on the non-linear force for one jointed link given by equation (35), the vector of non-linear force for all links in one truss unit is given by \( F_{NL} = c_d a \sin \omega t + c_d b \cos \omega t \), where \( c_d \) is the stiffness matrix of the non-linear truss unit, which is constructed by the stiffness of jointed link \( c_d \) calculated in step 3. By substituting \( c_d \) for \( K_L \) in equation (29), the stiffness matrix of an equivalent beam in jointed truss can be expressed as
where $K_{EB}$ is shown in equation (37). The stiffness coefficient matrix for the equivalent beam can be given.

- Using Newton–Raphson method, the correction value $\Delta q$ of displacement can be given. The iteration can not be ended until each element of $\Delta q$ is smaller than $\varepsilon$.

### Validation and discussion

#### The accuracy of the dynamic stiffness

The comparisons of the natural frequencies of a link calculated by a theoretical equation and dynamic stiffness are given to verify the relationship between the dynamic stiffness and the excitation frequency. The natural frequency of a link with one fixed end can be expressed as

$$
\omega_j = \frac{(2j - 1)\pi}{2l} \sqrt{\frac{E}{\rho}} \quad (j = 1, 2, \ldots)
$$

However, the displacement $D(\omega)$ can be obtained by giving the excitation frequency under unit excitation force.

$$
D(\omega) = 1/K_{L}(\omega)
$$

The parameters of the link are $E = 6.9 \times 10^{10}$ Pa, the width of the cross section $w_i = 0.03$ m, the height of the cross section $h_i = 0.03$ m, and the length $l = 2$ m. The displacements of the link change with excitation frequency on the basis of the assumption that $K_{L}(\omega) = |K_{el}(1,1)|$ or $|K_{el}(1,2)|$ in equation (12). The peak points of the displacements of the link correspond to the natural frequencies of the link. The natural frequencies calculated by $|K_{el}(1,1)|$ and $|K_{el}(1,2)|$ respectively are much close to each other as shown in Figure 7. Before $|K_{el}(1,1)|$ reaches its first maximum value, low excitation frequency does not change $|K_{el}(1,1)|$ obviously. Therefore, the first natural frequency of the link cannot be obtained by $|K_{el}(1,1)|$. However, they can express the stiffness change of a link with the excitation frequency.

The relative error of natural frequency can be expressed as

$$
R_\omega = \frac{\omega_i - \omega_j}{\omega_j} \times 100\%
$$

where $\omega_i$ is the natural frequency of the $j$th order calculated by the improved finite element.
The relative errors of the natural frequencies calculated by dynamic stiffness are less than 5% except the first order, as shown in Figure 8. Based on the wave propagation, the displacements of the two ends of link are out of sync. The displacements of the two ends are different with each other when the excitation frequency is low.

Dynamic analysis based on the improved finite element in wide frequency scope

The dynamic matrices can be used to get the natural frequencies of a link. The parameters of the link are the same as the settings in the previous section. When the element number of the link is given, the maximum order of the natural frequency for the link is constant. The relative errors of the natural frequencies, which are calculated by dynamic matrix (excitation frequency $\omega = 1$ Hz) and equation (39), are given in Figure 9. The relative error of the natural frequency increases to nearly 20% when the order is close to four-fifths of all orders, but decreases in the scope of the last one-fifth of total orders. The changes of relative error of the natural frequency with elements of the link are given in Figure 9(b). It is found that the error decreases with the number of elements. When the number of elements is not less than 10, the errors of the natural frequencies of the orders from 1 to 5 are smaller than 5%. But when the number of elements is not less than 20, the errors of the orders from 7 to 9 can be below 5%. So the best solution is decided by the number of both elements and natural-frequency orders. The higher of the order number of the natural frequency and the larger of the number of elements, the more accurate the solution is. Additionally, because the excitation frequency is low and close to the static state, the relative error of high order natural frequencies are higher than low order ones.

Effects of the excitation frequency on the natural frequency of the link are given to take better advantage of the matrices of stiffness and mass, as shown in Figure 10. The relative error of the natural frequency
in low order increases with the excitation frequency. When the excitation frequency increases from 1 Hz to $4 \times 10^5$ Hz, the order of the natural frequency with minimum error increases from the first order to the sixth order, as shown in Figure 10(b-I). However, the error of high-order frequency decreases with the excitation frequency. The errors of the orders following the one with minimum error decrease with the excitation frequency, such as the error of the 11th order which decreases from 11% to 6.9% shown in Figure 10(b-II). The error of the new element is equal to the common element when the excitation frequency is about or below 1 Hz. So, more accurate dynamic responses of trusses are obtained by using the common element in low excitation frequencies and the improved element in other frequencies.

The accuracy of the equivalent model

A plane truss with 10 truss units for analysis is shown in Figure 11. The calculation of the natural frequency of this plane truss introduces the improved matrices of the stiffness and the mass for a link. The parameters are shown in Table 1.

The natural frequencies of this type of truss with different number of truss units are calculated by equivalent links and software ANSYS.10 where the link of the truss is substituted by link 8 element. The results of the truss with different number of truss unit are compared with the results of the equivalent link shown in Figure 12. The out-of-plane equivalence is calculated based on the improved beam element. The results are compared with software results as shown in Figure 13.

The comparisons of the results show that the accuracy of the equivalent model can satisfy the need of the application of engineering. The errors of both the equivalent link for the plane truss and the equivalent beam for out-of-plane truss decrease with the number of truss units. Because the vibration mode of the truss includes not only purely bending, but also torsion, when the number of truss units is nine, the third order natural frequency for out-of-plane truss displays slight increase.

The calculation of a jointed structure

The simulation of a jointed structure with 10 periodic units as shown in Figure 14 is given to evaluate the

![Figure 11. Plane truss (n = 10).](image)

**Table 1. The parameters of the truss.**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus, $E$</td>
<td>$6.9 \times 10^{10}$ Pa</td>
<td></td>
</tr>
<tr>
<td>Density, $\rho$</td>
<td>$2.816 \times 10^3$ kg/m$^3$</td>
<td></td>
</tr>
<tr>
<td>Height and width of the truss unit, $l$</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>The cross section of the link in the truss, $h \times w$</td>
<td>$0.03 \times 0.03$ m $\times$ m</td>
<td></td>
</tr>
<tr>
<td>The horizontal angle of the diagonal link, $\alpha$</td>
<td>$\pi/4$ rad</td>
<td></td>
</tr>
</tbody>
</table>

![Figure 12. The comparison of the truss and the equivalent link in plane.](image)

![Figure 13. The comparison of the truss and the equivalent beam out-of-plane.](image)
Figure 14. The jointed structure with 10 periodic units.

Figure 15. The amplitude–frequency curve: (a) $K_3 = 0$; and (b) $K_3 = 10^4 K_L$.

Figure 16. The natural frequency of the jointed truss. (a) the first order of natural frequency, (b) the second order of natural frequency, (c) the third order of natural frequency and (d) the fourth order of natural frequency.
effects of joints on the non-linear structure, the parameters of which are shown in Table 1. Coefficients of the joint stiffness \( K_1 = K_L \) and \( K_3 = 0, 10^4 K_L \) and excitation forces \( F_i = 2^{i-3} \times 10^{-5} EA \) \((i = 1, 2, 3, 4)\) are set.

The amplitude–frequency curve is given to obtain the influence of joints on the jointed structure. The natural frequencies of the linear joint structure are shown in Figure 15(a). The first natural frequency of the linear joint structure is 10.66 Hz. The natural frequencies are not related to the excitation condition. The natural frequencies of the non-linear jointed structure are all larger than the linear joint structure shown in Figure 15(b). Because the coefficient of the cubic spring is greater than zero, the non-linear characteristic is close to that of hard springs. And, when \( K_3 = 10^4 K_L \) and the excitation force increases from \( F_1 \) to \( F_4 \), the first natural frequency of the jointed structure increases by 28.78%.

The first four natural frequencies of the jointed truss with different joint stiffness and excitation frequency are given in Figure 16, where \( C_{K_3} = K_3/K_L \).

### Table 2. The results of the fitting.

<table>
<thead>
<tr>
<th>Order of natural frequency</th>
<th>( \omega_1(K_3 = 0) )</th>
<th>( C_1 )</th>
<th>( C_2 \times 10^{-5} )</th>
<th>( C_3 )</th>
<th>RMSE (Hz)</th>
<th>RMSE/( \omega_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.66</td>
<td>1.22</td>
<td>2.31</td>
<td>7.68</td>
<td>0.26</td>
<td>2.43%</td>
</tr>
<tr>
<td>2</td>
<td>55.16</td>
<td>3.65</td>
<td>3.57</td>
<td>49.09</td>
<td>0.48</td>
<td>0.88%</td>
</tr>
<tr>
<td>3</td>
<td>129.70</td>
<td>8.90</td>
<td>4.22</td>
<td>120.71</td>
<td>0.83</td>
<td>0.64%</td>
</tr>
<tr>
<td>4</td>
<td>213.20</td>
<td>16.19</td>
<td>4.81</td>
<td>202.18</td>
<td>1.21</td>
<td>0.57%</td>
</tr>
</tbody>
</table>

RMSE: root mean square error.

---

**Figure 17.** The natural frequency of the jointed truss with exponential. (a) the first order of natural frequency, (b) the second order of natural frequency, (c) the third order of natural frequency and (d) the fourth order of natural frequency.
joint stiffness. When $C_{K3}$ is smaller than 1250, the first two natural frequencies almost increase linearly with the excitation force. While $C_{K3}$ is larger than 1250, the first two natural frequencies increase non-linearly with the excitation force. However, the third and forth natural frequencies almost increase linearly with the excitation frequency, no matter what the value is. When the excitation force is constant, the natural frequency increases exponentially with $C_{K3}$, as shown in Figure 16.

Therefore, the non-linear fitting of the natural frequencies can be expressed as

$$\omega_{ij} = C_1 \log_{10}(C_{Kj}) + C_2 F + C_3 \quad (42)$$

The coefficients of the fitting results of the first four natural frequencies are shown in Table 2. The coefficient $C_1$ and $C_2$ almost linearly increase with the natural frequency of linear jointed structure $\omega_{ij} (K_j = 0)$. The root mean square error (RMSE) of the first four natural frequencies of the jointed structure increases with $\omega_{ij}$. However, the RMSE-$\omega_{ij}$ ratio decreases with $\omega_{ij}$. The fitting and analysis results are shown in Figure 17. So the simulation of equivalent non-linear truss can get the dynamic characteristic of the jointed structure. This method can also be used for studying the effect of other characteristic of joints on jointed structures.

**Conclusions**

To get the linear and non-linear structure responses with high excitation frequency, this paper builds up the dynamic matrices of stiffness and mass for a link and a beam which are based on axial wave motion and transverse wave motion respectively. The accuracy of the dynamic matrices is evaluated when the number of elements and excitation frequencies change. The improved element is more accurate than common element when the excitation frequencies are relatively high.

To satisfy the analysis need of a large structure, the given equivalence of the truss is based on the strain energy and displacement equivalence. The equivalent model is accurate to calculate the large truss. When the number of truss units increases to 10, the error of the natural frequency of the periodic truss in plane is below 6%. When the number of truss units is or more than 6, the errors of the natural frequency of the truss out-of-plane are below 5%. Meanwhile, the error of the equivalent model decreases with the number of truss units. The equivalent jointed structure is also given for solving large deployable trusses. The two equivalent trusses can be calculated by the new element matrix.

Taking the cubic spring as an example, the non-linear characteristic of a jointed structure is given. The natural frequency of the jointed structure changes with coefficient of non-linear stiffness of the joint and the excitation force dramatically. The natural frequency of the structure with larger coefficients $K_1$ and $K_2$ increases more rapidly with increasing excitation force. The linear fitting function of the analysis results is given to evaluate the effects of joints on the jointed truss. The RMSE-$\omega_{ij}$ ratio is below 3%, and decreases with the natural frequency of the jointed truss.

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**Conflict of interest**

None declared.

**References**


Appendix 1

Notation

\( a, b \)
vector of expansion coefficient of \( d \) (4 \( \times \) 1)

\( a, b \)
expansion coefficient of \( d \) unspecified amplitudes of \( \bar{u} \)

\( a(\omega), d(\omega) \)
unspecified amplitudes of \( \bar{v} \)

\( A \)
cross section

\( A_d \)
coefficient matrix of displacement conversion

\( B_c \)
stiffness matrix of jointed truss unit (6 \( \times \) 6)

\( \sqrt{E/\rho} \)
stiffness of jointed link

\( e \)
vector of unspecified amplitudes of \( \bar{v} \), (4 \( \times \) 1)

\( e_{1}, e_{2}, e_{3}, e_{4} \)
coefficients of \( \omega_{1} \)

\( K_{1} \)
displacements of the jointed link, the joint, and the link displacement of the link with one fixed end

\( K_{1}/K \)
base of natural logarithm

\( L \)
link number in the truss unit

\( M \)
Young’s modulus

\( M_{NL} \)
axial force vector of the jointed link in the truss

\( (4 \times 1) \)
axial force of jointed link

\( F \)
Fourier-transformed axial force

\( F_{NL} \)
shear modulus

\( G \)
height of the cross section of the link in the jointed truss

\( h \)
imaginary part

\( i \)
inertia moment

\( k_{1} \)

\( K_{eI} \)
stiffness matrices of the beam and the link (4 \( \times \) 4), (2 \( \times \) 2)

\( K_{eI}(1,1), K_{eI}(1,2), K_{eI}(2,1), K_{eI}(2,2) \)
elements of \( K_{eI} \)

\( K_{eI}, K_{eL} \)
expanded stiffness matrix of the link (4 \( \times \) 4)

\( K_{eI} \)
stiffness matrices of the equivalent beam, the equivalent link, and the truss unit (6 \( \times \) 6), (4 \( \times \) 4), (8 \( \times \) 8)

\( K_{eL} \)
stiffness matrix of a truss linear and nonlinear stiffness of the joint, stiffness of the link in the jointed link length of link mass matrices of the beam and the link (4 \( \times \) 4), (2 \( \times \) 2)

\( k_{truss} \)
elements of \( M_{eI} \)
expanded mass matrix of the link (4 \( \times \) 4)

\( K_{truss} \)

\( n_{1}, n_{2}, n_{3}, n_{4} \)

\( M_{truss} \)

\( M_{truss}(1,1), M_{truss}(1,2), M_{truss}(2,1), M_{truss}(2,2) \)
elements of \( M_{truss} \)

\( M_{truss} \)

\( M_{truss}(1,1), M_{truss}(1,2), M_{truss}(2,1), M_{truss}(2,2) \)
elements of \( M_{truss} \)

\( M_{truss} \)

\( M_{truss}(1,1), M_{truss}(1,2), M_{truss}(2,1), M_{truss}(2,2) \)
elements of \( M_{truss} \)

\( M_{truss} \)
relation matrix of $V$ and $C$

$N_b$ coefficient vector of shape function of the beam $(1 \times 4)$

$N_B$ coefficient vector of shape function of the link $(1 \times 4)$

$q$ displacement vector of a truss unit $(8 \times 1)$

$q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8$ node displacements of a truss unit

$q_j$ displacement of the $j$th degree of freedom in a truss unit, $1 \leq j \leq 8$

$ar{Q}$ Fourier-transformed shear force

$R$ relative error of the natural frequency

$R_{\omega}$ natural frequency of the $j$th order of a link with one fixed end

$	ilde{t}$ natural frequency of linear jointed structure($K_3=0$)

$u, \tilde{u}$ axial displacement, Fourier-transformed axial displacement

$u_1, u_2$ displacements of link ends in $x$-direction

$\tilde{u}_1(0, \omega), \tilde{u}_2(l, \omega)$ Fourier-transformed of $u_1$ and $u_2$

$U_{EL}, U_T$ strain energy of the equivalent link and the truss unit

$v_1, v_2$ displacement vector of the equivalent link $(4 \times 1)$

$\tilde{v}_b, \tilde{v}_s$ displacements of the link in $y$-direction

$V$ Fourier-transformed transverse displacements produced by bending and shear

$V_{EL}, V_T$ Fourier-transformed displacement vector of the beam $(4 \times 1)$

$w$ kinetic energy of the equivalent link and the truss unit

$w$ width of the cross section of the link in the jointed truss

$W$ virtual strain energy

$x$ $x$-coordinate

$\alpha$ horizontal angle of the diagonal link in the jointed truss

$\Delta q$ iterative variation of $q$ $(8 \times 1)$

$\varepsilon$ permissible error values of $q$

$\eta$ $E/(\kappa G)$

$\kappa$ shear coefficient of the beam

$\omega$ excitation frequency

$\omega_j$ natural frequency of the $j$th order of a link with one fixed end

$\omega_L$ natural frequency of linear jointed structure($K_3=0$)

$\phi$ Fourier-transformed transverse rotation angle of the cross section

$\varphi$ phase angle

$\rho$ density

$\theta_1, \theta_2$ rotational degrees of freedoms at the ends of the beam

### Appendix 2

The expresses of the $N_b^{-1}$ can be expressed as

$$N_b^{-1}(1, 1) = 1/2(k_3(-2k_2 + e^{-ik_4 + k_2})(k_2 - k_3) + e^{-ik_4 - k_2}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(1, 2) = 1/2(2k_4 + e^{-ik_4 + k_2}(k_2 - k_3) - e^{-ik_4 - k_2}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(1, 3) = 1/2(k_3(-2k_2 e^{-ik_4} + e^{ik_3})(k_2 - k_3) + e^{-ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(1, 4) = 1/2(2k_4 e^{-ik_4} + e^{ik_3}(k_2 - k_3) - e^{-ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(2, 1) = 1/2(k_3(-2k_2 + e^{ik_4 + k_2})(k_2 - k_3) + e^{ik_4 - k_2}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(2, 2) = 1/2(k_3(-2k_2 e^{ik_4} + e^{-ik_3})(k_2 - k_3) + e^{ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(2, 3) = 1/2(k_3(-2k_2 e^{ik_4} + e^{-ik_3})(k_2 - k_3) + e^{ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(2, 4) = 1/2(k_3(-2k_2 e^{ik_4} - e^{-ik_3})(k_2 - k_3) + e^{ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(3, 1) = 1/2(k_3(-2k_3 e^{-ik_4 + k_2})(k_2 - k_3) + e^{ik_4 - k_2}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(3, 2) = 1/2(k_3(-2k_2 e^{-ik_4 + k_2})(k_2 - k_3) - e^{-ik_4 + k_2}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(3, 3) = 1/2(k_3(-2k_3 e^{-ik_4} + e^{ik_3})(k_2 - k_3) + e^{-ik_3}) \times (k_2 + k_3)/N_{b1}$$

$$N_b^{-1}(3, 4) = 1/2(k_3(-2k_2 e^{-ik_4} - e^{-ik_3})(k_2 - k_3) + e^{ik_3}) \times (k_2 + k_3)/N_{b1}$$
\[
N_b^{-1}(3, 4) = \frac{1}{2i}(2k_2e^{-ik_3} - e^{-ik_2}(k_2-k_3) - e^{ik_2} \times (k_2+k_3))/N_{b1}
\]

\[
N_b^{-1}(4, 1) = \frac{1}{2k_2}(-2k_3e^{i(k_3+k_2)}(k_2-k_3) + e^{-ik_2-k_3+k_2}) \times (k_2+k_3))/N_{b1}
\]

\[
N_b^{-1}(4, 2) = \frac{1}{2i}(-2k_2e^{i(k_3+k_2)}(k_2-k_3) + e^{-ik_2+k_3}) \times (k_2+k_3))/N_{b1}
\]

\[
N_b^{-1}(4, 3) = \frac{1}{2k_2}(-2k_3e^{i(k_3)} - e^{-ik_2}(k_2-k_3) + e^{ik_2} \times (k_2+k_3))/N_{b1} N_{b1}^{-1}(4, 4)
\]

\[
= \frac{1}{2i}(-2k_2e^{i(k_3+k_2)}+e^{-ik_2}(k_2-k_3) + e^{ik_2}(k_2+k_3))/N_{b1}
\]

where:

\[
N_{b1} = \frac{(\cos(l-k_3+k_2))(k_2 + k_3)^2 - \cos(lk_3+k_2))(k_2 - k_3)^2 - 4k_2k_3}{(k_2+2k_3)^2}
\]